Structure of Loss-sensitive Combinations of Three Missing Observations in Central Composite Designs

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Abstract:  
Different structures of loss-sensitive combinations in case of three missing observations in central composite designs have been studied in detail. The criteria for larger losses of three missing observations are different for different combinations and depends on k, α and position of the missing point on the surface. Particularly these four categories of missing combinations of three observations fff, fff, faoa and ada produces high losses. The empirical and graphical study has shown that r is minimum for different values of α and is equivalent to the variance of estimated response with different values of k and n.

Keywords: Central composite designs (CCD), missing observations, optimality criteria, minimum variance and formation of combinations.

1. Introduction
A central composite design consists of three parts:  
   i) a complete or fractional replication of \( n_f = 2^{k-p} \) factorial design points (f) with k factors of the form \((\pm 1, ..., \pm 1)\) usually called "cube" and \( p = 0 \) for complete factorial part.
   
   ii) a set of \( n_a=2k \) axial points (a) with coordinates \((\pm \alpha, 0, ..., 0), (0, \pm \alpha, 0, ..., 0), (0, ..., 0, \pm \alpha)\), usually called a "star". Where \( \alpha \) is the distance of the axial points from the design centre.

   iii) a set of \( n_c \) centre points (c) at \((0, ..., 0)\).

For the reason beyond experimenter's control, one cannot avoid the possibility of missing few observations in experimentation. In case of three missing observation, the losses are too high that may lead to breakdown the design. We must examine the formation and the effect of three missing observations to check the properties like orthogonality, balance and robustness of the design. Through empirical study, it is observed that the losses of these combinations of three missing observations from \( n = n_f + n_a + n_c \) design points fall into different groups of same losses. The number of groups depends on the design points (complete or half replication) and number of variables (k) in the chosen model. Ten categories of the combinations are formed from the factorial, axial and centre points when a total of three observations are missed from \( n_f, n_a \) and \( n_c \) design points.
Table 1 gives a summary of the groups of the same losses formed for various values of k for each of the 10 categories of three missing points.

<table>
<thead>
<tr>
<th>Combination of missing three design points</th>
<th>The losses of missing Combinations of each category fall into no. of groups for</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>k=2</td>
</tr>
<tr>
<td><em><strong>ccc</strong></em></td>
<td>1</td>
</tr>
<tr>
<td><em><strong>ace</strong></em></td>
<td>1</td>
</tr>
<tr>
<td><em><strong>hec</strong></em></td>
<td>1</td>
</tr>
<tr>
<td><em><strong>fac</strong></em></td>
<td>2</td>
</tr>
<tr>
<td><em><strong>aac</strong></em></td>
<td>2</td>
</tr>
<tr>
<td><em><strong>aaa</strong></em></td>
<td>1</td>
</tr>
<tr>
<td><em><strong>faa</strong></em></td>
<td>4</td>
</tr>
<tr>
<td><em><strong>ffc</strong></em></td>
<td>k</td>
</tr>
<tr>
<td><em><strong>ffa</strong></em></td>
<td>3k-2</td>
</tr>
<tr>
<td><em><strong>fff</strong></em></td>
<td>k(k-1)/2</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>( \frac{(k^2+7k+20)}{2} )</td>
</tr>
</tbody>
</table>

In section 2, we have studied in detail the different structures of the combinations of three missing observations for k=2 to 6 factors in CCD. Section 3 contains design with half replication of factorial part and complete replication of axial part. Section 4 analyzes structure of losses based on minimum-variance criterion. Section 5 concludes with final remarks and discussion.

2. **Structure of loss-sensitive combinations of three missing observations**

In this section, we will investigate the losses \( \lambda_{wuv} \) due to a set of three missing observations (where \( u=1, ..., n-2, v=2, ..., n-1 \) and \( w=3, ..., n \) within all the groups for a variety of different configurations for \( k=2, ..., 6 \). All the combinations of three missing observations do not produce same losses \( \lambda_{wuv} \). Some combinations of three missing observations have higher losses than the other combinations. The size of these losses of the different combinations of three missing observations depends on the following:

(i) The signs of the variables of the missing point i.e. the position of the missing points on the surface,
(ii) The size of \( \alpha \),
(iii) The number of variables \( k \),
(iv) The number of design points, and
(v) The proportion of the factorial \( \left( \frac{n_r}{n} \right) \) and axial points \( \left( \frac{n_{av}}{n} \right) \) for fixed \( n \), centre points.
Particularly the four categories of the missing combinations of three observations \( fff, ffa, faa, \) and \( aea \) produces high losses. The trend of the losses of these combinations changes with the change in \( k, \alpha \) and the proportion of the factorial and axial points. Detailed study of the formation of the combinations of three missing observations was done for each \( k \) but the same is being presented for \( k=4 \) (the middle of the range) only because of the space limitation. It has been found that the criteria for larger losses of three missing observations is different for \( k=2 \) and \( k=3 \) to 6.

2.1 High losses combinations for \( k=2 \)

In CCD with \( k=2 \), the loss of a combination due to three missing observations that are proximate (closer) to each other (excluding centre points) have larger value of loss as compared to all other combinations of three missing observations of same configuration. Akram (2002, Ch 3) observed that the maximum loss of a combination of three missing observations is only due to the combination \((ffa)\) of two factorial and one axial points or the combination \((faa)\) with one factorial and two axial points becomes collinear, approaches 1.0 and the CCD breaks down. For example

\[
\begin{array}{ccc}
\text{Point No.} & x_1 & x_2 \\
f(2) & 1 & -1 \\
f(4) & 1 & 1 \\
a(5) & \alpha & 0
\end{array} \quad \begin{array}{ccc}
\text{Point No.} & x_1 & x_2 \\
f(1) & -1 & -1 \\
f(3) & -1 & 1 \\
a(6) & -\alpha & 0
\end{array} \quad \begin{array}{ccc}
\text{Point No.} & x_1 & x_2 \\
f(2) & 1 & -1 \\
\ldots \ldots \\
a(8) & 0 & -\alpha
\end{array}
\]

etc.

Now, considering another category of combinations \((fff)\) of three missing factorial observations which produces same losses but the losses will be higher only when \( \alpha=1.0 \). Same is true for \( aea \), because the triangles of all the combinations of three missing axial points from \( n_a=4 \) have equal angles. Thus the distance between these three points is same and produces high losses for large value of \( \alpha \).

2.2 High losses combinations for \( k=3 \) to 6

Akram (2002, Ch 3) through empirical study shows that all the combinations of missing three factorial points ‘\( fff \)’ fall into \( k(k-1)/2 \) groups for \( k=3, 4, 5 \) and \( (k(k-1)+2)/2 \) for \( k=6 \) with same loss of efficiency. Out of these combinations, one group of the combinations of missing three factorial points has high losses. The values of the losses depend on the signs of the variables corresponding to the position of the missing points in the combination. The maximum loss of missing three factorial points is obtained, when the signs of variables \( (x_1, x_2, ..., x_k) \) corresponding to the two factorial points \( ff \) are complementary i.e. the sum of the two row-vectors of the design matrix \( D \) corresponding to these two points is a zero-vector, and the third factorial point \( f \) is the nearest to any one of the two factorial points in sense of signs.

Let \( X_{f(1)} = [-1 \quad -1 \quad -1 \quad -1] \) is the first row-vector and \( X_{f(16)} = [1 \quad 1 \quad 1 \quad 1] \) is the second row-vector of the following matrix of three missing observations. The sum of \( X_{f(1)} \) and \( X_{f(16)} \) is the zero vector i.e. \( X_{f(1)} + X_{f(16)} = 0 \). The following combinations produce maximum loss for missing three factorial points when \( k=4 \), viz.
Structure of Loss-sensitive Combinations of Three Missing ...

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
1 & 1 & -1 & -1 & -1 \\
2 & -1 & 1 & -1 & -1 \\
3 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
5 & 1 & -1 & -1 & 1 \\
6 & -1 & 1 & -1 & 1 \\
7 & 1 & 1 & 1 & -1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
2 & 1 & -1 & -1 & 1 \\
3 & -1 & 1 & -1 & 1 \\
4 & 1 & 1 & 1 & -1 \\
\end{array}
\]

Now we will consider the combinations of three missing observations with two factorial and one axial point. All the combinations of \( ffa \) whose losses fall into \((3k-2)\) groups. In these combinations of \( ffa \), one particular group of \( ffa \) has very high losses for smaller values of \( k \) and \( \alpha \). These losses also depend on the signs of the \( k \) variables in the row corresponding to the missing points. The loss of the combination corresponding to the two missing factorial points \( 'ff' \) with any third axial point \( 'a' \) is very large, when the signs of \( k \) variables \( (x_1, x_2, \ldots, x_k) \) corresponding to the two missing factorial points \( 'ff' \) of the design matrix \( D \) are complementary. Thus, there are \( k \times 2^k \) combinations of \( ffa \) of this category which have the larger losses than remaining combinations of \( ffa \).

For \( k = 4 \),

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
1 & 1 & 1 & 1 & 1 \\
2 & 1 & -1 & -1 & -1 \\
3 & -1 & 1 & -1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
5 & -1 & -1 & -1 & 1 \\
6 & -1 & 1 & 1 & 1 \\
7 & 1 & 1 & 1 & -1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
8 & 1 & -1 & -1 & -1 \\
9 & -1 & 1 & -1 & 1 \\
10 & 1 & 1 & 1 & -1 \\
\end{array}
\]

The losses of all the combinations of missing one factorial and two axial points fall into four groups with same losses of efficiency. It has been observed that the loss becomes very large, when a combination of two axial points \( aa \) belong to any one variable among \( k \) variables \( (x_1, x_2, \ldots, x_k) \) with any third factorial point \( 'f' \) is missing.

There are \( k \times 2^k \) combinations \( ffa \) of this type for \( n_f = 2^k \) and \( n_a = 2k \) points. These combinations produce higher losses among all the combinations of same configurations particularly for larger values of \( k \). For example

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
7 & 1 & -1 & 1 & 1 \\
17 & 1 & 0 & 0 & 0 \\
18 & -1 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
5 & -1 & 1 & -1 & 1 \\
23 & 0 & 0 & 0 & \alpha \\
24 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
15 & 1 & -1 & 1 & 1 \\
21 & 0 & 0 & \alpha & 0 \\
22 & 0 & 0 & 0 & -\alpha \\
\end{array}
\]

The losses of all the combinations of three missing observations from the axial part of \( n_a = 2k \) points fall into two groups with similar losses. These two groups consist of the following combinations:

i) \( 4k \) (\( k-1 \)) \( (k-2)/3 \) combinations of three missing axial points belonging to the different three variables out of \( k \) variables (i.e. lie on the different axis) fall in the first group.

ii) There are \( 2k(k-1) \) combinations of two axial points \( aa \) belonging to the same variable (or lie on the same axis) among the \( k \) variable \( (x_1, x_2, \ldots, x_k) \) in the chosen model with any third axial point belonging to the remaining \( (k-1) \) variables have the larger losses than all the other combinations of three axial
points \( aaa \) from different variables. Especially this type combination is very damaging for larger values of \( k \) and \( \alpha \). When \( k \geq 5 \), the combinations of this category have the largest loss among all types of combinations of three missing observations of \( n \) design points for complete replication of factorial and axial part. This is due to smaller number of axial points as compared to factorial points in the design with \( k \geq 5 \).

For example, this combination may be illustrated as

\[
\begin{array}{c|cccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 \\
\hline
\sigma(17) & \alpha & 0 & 0 & 0 \\
\sigma(18) & -\alpha & 0 & 0 & 0 \\
\sigma(19) & 0 & \alpha & 0 & 0 \\
\end{array}
\]

Lastly, we will consider the combinations of three missing observations with two coaxial and one-centre points. Akram (2002, Ch: 3) observed that the combinations of two axial points \( aaa \) belonging to the same variable with any third design point have the larger loss among the maximum losses. It also observed that the missing combination with two coaxial and one-centre points have the third maximum loss for \( k = 5, 6 \).

3. Design with half replication of factorial part and complete replication of axial part

The loss pattern of combinations of three missing observations is different from the complete replication of factorial part. Akram (2002, Ch: 3) observed that the combination of three missing observations consisting of one, two or three factorial point(s) has higher losses as compared to their complete replication counterpart. It is also found that the factorial part is more loss sensitive than the axial part for certain range of \( \alpha \), particularly when two factorial points are already missing. Table 2 gives the condensed information about number of groups \( 4(k+1) \) for half replication of factorial part and complete replication of axial part, in which the losses of all possible combinations of three missing observations fall.

### Table 2: Summary of the formed groups for different values of \( k \) with half replication of factorial part

<table>
<thead>
<tr>
<th>Combinations of design points</th>
<th>Losses of the combinations fall into no. of groups</th>
<th>No. of groups</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>for ( k = 5 )</td>
<td>for ( k = 6 )</td>
</tr>
<tr>
<td>ccc</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>acc</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>fcc</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>fac</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>aac</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>aaa</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>faa</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>ffc</td>
<td>( k-3 )</td>
<td>2</td>
</tr>
<tr>
<td>ffa</td>
<td>( K+1 )</td>
<td>6</td>
</tr>
<tr>
<td>fff</td>
<td>( 2k-7 )</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>( 4(k+1) )</td>
<td>24</td>
</tr>
</tbody>
</table>
For \( k=5 \), the loss of missing combination is very high for this configuration \( ffa \). The missing of this particular combination makes the design breakdown i.e. it produces loss \( =1 \) for \( \alpha =1.0 \). This situation may get improved by increasing \( \alpha \) values.

For example the formation of these combinations as follows,

\[
\begin{array}{cccccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 & x_5 \\
 f(1) & 1 & -1 & -1 & -1 & -1 \\
 f(8) & -1 & 1 & 1 & 1 & -1 \\
 a(26) & 0 & 0 & 0 & 0 & -\alpha \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{Point No.} & x_1 & x_2 & x_3 & x_4 & x_5 \\
 f(9) & -1 & -1 & -1 & -1 & -1 \\
 f(15) & -1 & -1 & 1 & 1 & 1 \\
 a(18) & -\alpha & 0 & 0 & 0 & 0 \\
\end{array}
\]

4. **Structure of losses based on minimum variance criterion**

For the response surface model is \( \mathbf{Y} = \mathbf{X} \beta + \mathbf{e} \), with \( \hat{\mathbf{y}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{R}\hat{\mathbf{y}} \) and \( \mathbf{R} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \) matrix, where \( \hat{\mathbf{y}} \) is a vector of predicted values and \( \mathbf{y} \) is a vector of observed values.

Box and Draper (1975) studied the change in predicted response \( \hat{y}_i \) due to \( c \) change in observed value \( y_i \) will be \( \delta_i = cr_{iu} \) where \( (i, u)=1, \ldots, n \). The discrepancy in the predicted values of \( \hat{y}_1, \hat{y}_2, \ldots \) is \( \delta = (\delta_1, \delta_2, \ldots, \delta_n)' \) and \( \hat{\delta} \) is the vector of such discrepancies. One can express overall discrepancy \( d_u \) caused by the aberration \( c \) in the \( u \)th observation

\[
y_u = d_u = \delta' \delta = \sum_{i=1}^{n} \delta_i^2 = [cr_{iu} \quad \ldots \quad cr_{mu}] [c r_{iu} \ldots c r_{mu}] = c^2 \sum_{i=1}^{n} r_{iu}^2 = c^2 r_{uu}
\]

Where \( r_{uu} = \sum_{i=1}^{n} r_{iu}^2 \) because \( \mathbf{R} \) is symmetric and idempotent matrix and

\[
\bar{d} = \frac{c^2 \sum r_{uu}}{n} = \frac{c^2 p}{n}
\]

\[
V(d) = \frac{\sum (d_u - \bar{d})^2}{n} = \frac{(\sum d_u^2 - n \bar{d}^2)}{n} = \frac{(c^4 \sum r_{uu}^2 - n c^4 p^2 / n^2)}{n} = c^4 (r - p^2 / n) / n,
\]

Where \( tr(R) = tr(R^2) = \text{Rank}(R) = p \) and \( r = \sum_{i=1}^{n} r_{uu}^2 \). To make sure of insensitivity to missing observations or outliers, \( Var(d) \) should be minimized which is equivalent to minimization of \( r \) for fixed \( n \) and \( p \). Consider the variance of predicted value \( \hat{y}_u \) at the \( u \)th design point that is \( V_u = V(\hat{y}_u) = r_{uu} \sigma^2 \) and also \( V(\hat{\mathbf{y}}) = \mathbf{R} \sigma^2 \). Box and Draper
(1975) postulated that we should choose a design to make all the $v_u = r_{uu} \sigma^2$ (where $u=1, 2, ..., n$) as similar as possible that is equivalent to minimize $r_{uu}$.

$$V = \frac{\sum (v_u - \bar{v})^2}{n}, \quad V/\sigma^4 = \frac{(r - \mu)^2}{n}$$

This shows that the criterion of minimization of $V$, $V(d)$ or ‘r’ are all equivalent. Andrews and Herzberg (1979) observed that smaller the value of the sample variances $V$ of predicted responses better (or more robust) the design is. Akhtar (1985) also gave the relationship of the diagonal values $r_{uu}$ of $R = X(X'X)^{-1}X'$ matrix for three missing observations,

$$A_{uvw} = [(1 - r_{uu})(1 - r_{vw})(1 - r_{uw}) - r_{uv}^2(1 - r_{vu}) - r_{uw}^2(1 - r_{wu}) - r_{vu}^2(1 - r_{uv}) - 2 r_{uu} r_{vw} r_{uw}]$$

is the diagonal element of the third compound of the (I-R). The diagonal element $A_{uvw}$ gives an exact proportion of reduction in $|X'X|$ caused by the loss of a combination of three missing observations. The diagonal elements of R matrix corresponding to factorial, axial and centre points are represented by $r_f$, $r_a$ and $r_c$ respectively.

i.e. diag $(R) = (r_f, ..., r_f, r_a, ..., r_a, r_c, ..., r_c)$

There are $n_f$ row vectors of factorial part, $n_v$ row vectors of axial part and $n_c$ row vectors of centre part. After some mathematical work, their values can be found as:

$$r_f = \{(4n_f(k^2 - k + 2) + k(k - 1)n_c)\alpha^6 + 2n_f(n_f^{-1} + k - k + 2) - 4k(k^2 - k + 2)$$
$$+ 2n_fk(k^2 - k + 2) - 2n_f(k^2 - k + 2) + 2(k^2 - k + 2) + (k^2 - k + 2)n_c \alpha^2$$
$$+ kn_f[2k^2 - (k + 2)] + (k^2 + k + 2)n_c)/\{8(n_f + n_c)\alpha^6 + 4n_f(n_f + n_c) - 4k)\alpha^4$$
$$+ 4kn_f(2k - 2n_f + n_c)\alpha^2 + 2kn_f^2(2k + n_c)\}$$

$$r_a = \{8(n_f + n_c)\alpha^6 + 2n_f(n_f + n_c) - 8k)\alpha^4 + 2n_f[2k(k - n_f) + (2k - 1)n_c] \alpha^2$$
$$+ n_f^2[2k^2 + (k - 1)n_c))/\{8(n_f + n_c)\alpha^6 + 4n_f(n_f + n_c) - 4k)\alpha^4$$
$$+ 4kn_f(2k - 2n_f + n_c)\alpha^2 + 2kn_f^2(2k + n_c)\}$$

$$r_c = \frac{\left(n_c + 2n_f(k - \alpha^2)^2\right)}{(2\alpha^4 + kn_f)}.$$
matrix \( R = X(X'X)^{-1}X' \). The values of alpha (\( \alpha \)) are determined for CCD at different counts of \( n_c \) (centre points) by using the above equation. Empirically and graphically minimum values of \( r \) are determined at different values of \( n_c \) \( \alpha \) and has been shown in the Table 3. It has been pointed out that we need a minimum of \( n_c = 4 \), because we are dealing with the situation of three missing observations. We studied the case for \( n_c = 1, 2, \ldots, 6 \) to have a better assessment of the trend around \( n_c = 4 \).

**Table 3:** Values of \( \alpha \) giving minimum of \( r = \sum r_{uu}^2 \) at different values of \( n_c \) with \( n_f = 2^k \) and \( n_m = 2k \).

<table>
<thead>
<tr>
<th>No. of Centre Points ( n_c )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K=2 )</td>
<td>Min. ( r )</td>
<td>4.07054</td>
<td>3.625</td>
<td>3.4583</td>
<td>3.375</td>
<td>3.325</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1.16889</td>
<td>1.4142</td>
<td>1.41431</td>
<td>1.4143</td>
<td>1.41421</td>
<td>1.41417</td>
</tr>
<tr>
<td>( K=3 )</td>
<td>Min. ( r )</td>
<td>6.69051</td>
<td>6.28884</td>
<td>6.12271</td>
<td>6.03931</td>
<td>5.98916</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1.66144</td>
<td>1.78021</td>
<td>1.7712</td>
<td>1.77243</td>
<td>1.77496</td>
<td>1.7777</td>
</tr>
<tr>
<td>( K=4 )</td>
<td>Min. ( r )</td>
<td>0.01363</td>
<td>8.66667</td>
<td>8.5</td>
<td>8.41667</td>
<td>8.36667</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1.66144</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( K=5 )</td>
<td>Min. ( r )</td>
<td>10.2933</td>
<td>10.0584</td>
<td>9.90478</td>
<td>9.82213</td>
<td>9.77169</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1.79957</td>
<td>2.03253</td>
<td>2.12652</td>
<td>2.13343</td>
<td>2.12882</td>
<td>2.12127</td>
</tr>
<tr>
<td>( K=6 )</td>
<td>Min. ( r )</td>
<td>10.492</td>
<td>10.3586</td>
<td>10.2276</td>
<td>10.148</td>
<td>10.0971</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.00014</td>
<td>2.12077</td>
<td>2.23071</td>
<td>2.25034</td>
<td>2.24369</td>
<td>2.22912</td>
</tr>
</tbody>
</table>

**Conclusion and Final remarks**

After empirical study, we have concluded that when complete replication of factorial point is used for \( k=2, 3 \) or \( 4 \) with smaller values of \( \alpha \) factorial points are more loss sensitive but for larger values \( \alpha \), axial points become more loss sensitive, the axial points remain loss sensitive for \( k=5, 6 \) irrespective of \( \alpha \) respectively. On the other hand with half replication of factorial part, factorial points remain loss sensitive for \( k=5, 6 \). The loss sensitivity pattern depends on the proportion of axial and factorial points in the design, \( \alpha \) and the position of the missing point in the combination. The triangle of combination of three missing observations that are proximate (closer) to each other (excluding centre points) have larger value of loss than all the other combinations of three missing observations of same configuration for \( k=2 \).

**Figure 1:** Graphical representation of Minimum \( r \) with \( k=4, n_f=16, n_m=8 \) at different values of \( n_c \).

\[
 r = \sum r_{uu}^2
\]
References


